

Abstract:

The traditional way of assessing mathematics learning in a student or the progress in his ability to think and reason has been through the use of written tests. Correct answers given in these tests point out mastery on part of the student while wrong answers hint at deficiency. Some alternatives to this method of assessment are offered implying a recasting of the basis for teaching and learning so that they go hand in hand with growth in mathematical thinking and power.

Full Text :

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Two of the first three Standards for Teaching Mathematics begin with the following statements:

The goal of teaching mathematics is to help all students develop mathematical power (emphasis added) and all students can learn to think mathematically (emphasis added) (NCTM 1991).

What exactly is "mathematical power" to someone who has always identified mathematics as being the mastery of facts, such as the multiplication tables, and procedures, such as the long division algorithm? What does it mean to "think mathematically" to a teacher who always struggled with story problems as a student? To those teachers who fit these descriptions, and a sizable number do, assessing students' mathematical power and mathematical thinking is even more bewildering.

Piaget refers to mathematical power as having to do with acquiring personal "autonomy" (Kamii 1984). In this context, mathematical power is the student's ability to think and function independently from the teacher. Mathematical power is attained by helping students develop thought processes that can be used to solve problems and to determine whether solutions are appropriate. Mathematical power is gained by minimizing the student's dependence on the teacher or answer key.

Mathematical thinking involves, among other things, the abilities to (a) recognize patterns, (b) generalize common problem situations, (c) identify errors, and (d) generate alternative strategies. Mathematical thinking implies a systematic approach to quantitative problems. It is a by-product of learning and doing mathematics. At the same time, it can be the focus of learning rather than just a by-product. This focus suggests that all mathematics lessons could benefit by monitoring not only the content being studied but the actual growth in the student's ability to think and reason.

Until recently, the means by which teachers have assessed academic progress has been through the use of paper-and-pencil tests. The assumption has been that correct answers mean mastery; wrong answers mean learning deficiencies. Offered here are some ideas and alternatives to the more traditional methods of academic assessment. The suggested procedures, however, require a

re-thinking of the criteria for teaching and learning so that they are better aligned with the development of mathematical thinking and power. The article begins by offering seven such learning criteria. Included in the listing are suggestions for teaching and assessing students' progress that are compatible with each learning criterion. This discussion is followed with suggestions for assessing students' growth and grading it. Pedagogical foundations that support this approach and several content-specific applications of the methods outlined in this article can be found in Greenwood (1991, 1992).

Learning Criteria for Mathematical Thinking

The criteria found in table 1 are designed to be introduced to the students during the first class period to familiarize them with the notions of "mathematical thinking" and "mathematical power." They can be written on large posterboard and hung on the wall for the duration of the class. The idea is to refer to them often, daily if possible. Introducing the criteria during the first week of school helps students anticipate probing questions throughout the year because they begin to see from the start that more will be expected of them than just correct answers.

In the discussion that follows, the learning criteria themselves are enumerated. The statement that follows each criterion is the rationale that is offered to the students during the first meeting. The accompanying lesson example is given to help convey ideas pertaining to teaching and monitoring each criterion. Before turning to the task of illustrating the criteria, a word is in order about the nature of the examples used. Care has been taken to depict scenarios that cover a wide range of instructional situations from basic computational exercises to more general problem solving. The intent of these examples is to show the applicability of the criteria across this range and to assist the audience most in need of these ideas.

1. Everything you do in mathematics should make sense to you.

You should be able to explain your strategies and thoughts so they are clearly understood by others, not just repeat the steps of what to do to get an answer. The only way I can determine whether something makes sense to you is by listening to you explain it to me or to others. My assumption is that you can't explain something unless you understand it yourself. Sometimes you will be asked to explain how you worked a problem by using manipulatives, diagrams, patterns, or other materials we will be using. Eventually you will be asked to describe your thoughts and the pictures you have in your mind. So you see, you will have many opportunities to show whether something makes sense to you.

Discussion and example. Imagine walking around the room as students are working on mathematics problems where they are asked to draw a rectangle with a perimeter of 18 and a length of 4. You stop to observe Doris, who seems to be progressing well. You interrupt her work quietly and ask her to explain how she arrived at the drawing of a 4X5 rectangle. Her first response is to erase the answer and proceed to work the problem over. Does this scenario sound familiar? In many classrooms, students have acquired the notion that the teacher asks for explanations only for incorrect answers (Kamii 1984). To develop thinking skills, we must begin to value students' explanations for all work, all thinking--correct and incorrect alike. When students begin to see that such questions as "Are you willing to show me how you got that

answer?" "How do you know?" and "Why is that so?" apply to all the mathematics they do, they will develop more confidence in their original solutions and not feel compelled to erase them so quickly. Students who react to such questions with confidence illustrate growth in this criterion.

2. Whenever you get stuck, you should be able to use what you know to get yourself unstuck.

For you to learn to think for yourself, it is important that you learn how to answer your own questions. I'm asking you to learn how to build on what you know without asking the teacher or checking the answer key. To do so, it is important that you learn how to use what you know to straighten yourself out when you run into trouble. In the beginning, I will help you out by asking you some questions that will lead you to get unstuck. Later on, you will be able to ask the questions of yourself. When you start doing that, we will both know you are learning how to get unstuck.

Discussion and example. Rafael comes to you and asks, "What is 8×7 ? I forgot." To supply the answer (56) is to get the student "unstuck." Often it is seen as the thing to do. Because of short-term gains, supplying the answer saves time, it keeps the student on task, it minimizes errors, and the student is more apt to seek you out for help when it is needed. However, the long-term effects of consistently giving answers can hinder students' ability to learn what is needed to get themselves unstuck (Bruner 1968). Therefore, this criterion suggests that students' questions be answered with questions that will lead to getting unstuck. In this example, an appropriate response might be, "Do you know what 8×8 is?" If the student answers "64," the teacher then asks, "Can you use that information to figure out what 8×7 is?" If the student doesn't know what 8×8 is or can't use the answer to figure out 8×7 , try a follow-up question, such as "Do you know what 7×7 is, and if so can you use that information to figure out what 8×7 is?" As students anticipate your questions to their questions, they begin to ask the same kinds of questions of themselves. By the teacher's example they learn some strategies for getting unstuck. Students too can serve as "strategy teachers." They can share with the class difficulties they've encountered and tactics they've used to get unstuck. Sharing this information offers students alternatives and fosters independence in thinking.

3. You should be able to identify errors in answers, in the use of materials, and in thinking.

You can learn a great deal from your errors if you are willing to think about them once they've been spotted. At times I will ask whether you disagree with something and whether you can spot some mistake. If something is wrong and you can spot it, that will tell us both that you know how it is supposed to be. If nothing is wrong and you think there is, that will tell us something too. It will be important to listen to other students because I'm interested in how well you can help each other. We learn best when we take advantage of our mistakes because mistakes tell us a lot about what we need to work on. In a lot of ways, "mistakes" are stepping stones to learning.

Discussion and example. The class is using the fact that the sum of the measures of the three angles of any triangle is 180 degrees to find the sum of the measures of the four angles of any quadrilateral. Miguel has volunteered to come to the front of the class to demonstrate his answer of 720 degrees. He uses figure 1 to illustrate his thinking. He explains, "The quadrilateral ABCD is broken into four triangles, and each triangle has a total of 180 degrees. Therefore, the figure

ABCD has $4 \times 180^\circ = 720^\circ$ altogether."

Several students nod approvingly, and several others look confused. The teacher asks if any students disagree, and although none signal the recognition of an error, Maria asks to share her result, which seems to suggest a different answer. She illustrates her answer with figure 2. She explains, "I broke ABCD into two triangles, and therefore the figure ABCD has $2 \times 180^\circ = 360^\circ$ altogether."

The discussion that ensues can go several ways. The teacher can, of course, point out that none of the angle measures a, b, c, d, in figure 3 contribute to the sum of the measures of the angles of ABCD. This observation suggests that Miguel's answer needs to be decreased by 360 degrees, and the adjusted answer of $720^\circ - 360^\circ = 360^\circ$ agrees with Maria's. However, long-term advantages are gained by asking the students to discover the nature of the error in the first solution themselves (Bruner 1968; Wirtz 1976; Greenwood 1981). This criteria focuses on the importance of helping students develop the means and the confidence to question answers. Eventually, as students become more familiar with identifying errors, they also become more comfortable and articulate in demonstrating their thinking strategies.

4. Whenever you do a computation, you should use a minimum of counting.

I will be interested in how well you can learn to think about numbers and how you can picture them being added, subtracted, multiplied, and divided. To help you concentrate on thinking, I'm going to be asking you to try not to count. What I would like you to do, as much as possible, is think your way through a computation. By doing so, you'll actually find more and more patterns that will shorten your computations and make them easier for you to do. Every time you do a lot of counting, you actually stop thinking and begin to rely on the counting process itself. If you do this enough, you stop improving, you stop growing; you just fall back to counting. The more you count, the less you think. The more you think, the less you have to count. As we move along in our work together, I'm going to ask you to explain your work and your thinking in ways that minimize counting.

Discussion and example. This criterion is specifically meant to address the practice of counting on one's fingers to arrive at some addition or subtraction fact. Imagine the following situation occurring in a classroom. The teacher asks Marty, "What's eight plus five?" Marty is observed to use his fingers to count on from eight, moving a finger with each word, saying, "Nine, ten, eleven, twelve, thirteen. The answer is thirteen." Next imagine that twenty minutes have passed, and the teacher again asks Marty to find the answer to "eight plus five." It is not unusual to observe Marty repeat the earlier process in coming up with the answer of thirteen. In fact, when Marty is faced again with the same question, a day, week, or year later, it is not surprising that he employs the same process. The fact that counting always leads to the correct answer can actually hinder the student's learning anything permanent about the way numbers are combined and partitioned. This idea led Wirtz to conclude that "counting confounds thinking" and that "thinking is maximized when counting is minimized" (Wirtz 1976). It has also been shown that when students have already become dependent on counting as a means for producing addition and subtraction answers, it is counterproductive to demand that they discontinue the practice without helping them develop an alternative strategy with which to replace it (Newton 1985;

Greenwood 1981). Therefore, whenever counting is observed, the teacher can wait until the student is finished before asking, "I noticed how you got your answer, Marty. Could you have found it without having to count?" Additional questions posed to the class to offer alternatives could also be framed in this way: "Can someone think of another way you could figure out the answer without having to count?" As these questions are asked over time and as alternative strategies are generated by their peers, students begin to value noncounting strategies and thus begin to generate and employ thinking processes to replace counting practices.

5. You should be able to perform calculations with a minimum of rote pencil-paper computations.

This criterion is very similar to item 4, but it is included to give extra emphasis to mental-computational strategies and the use of calculators and computers. It is important that we explore a lot of different ways of solving arithmetic computations and become comfortable with several that go beyond paper and pencil. Exploring alternatives will also help us learn to think mathematically because we'll be concentrating on how things fit together.

Discussion and example. Consider the following set of number sentences:

$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

When each number sentence is presented one at a time, the teacher pauses and asks the class if each one is true. The first two sentences are easily judged by most classes as being true. The next sentence, however, oftentimes has students reaching for their pencils and calculators. In such situations, when students are asked to perform a computation in the context of solving a larger problem or investigating a pattern, it is important to take the opportunity to focus on mental-computational techniques. Thus, this criterion provides a forum around which the teacher can ask, "Can you determine whether the sentence is true by just using your mind?" Students will propose such strategies as "On the left side, '9 + 11 = 20,' and the other two numbers add up to '22,' and '20 + 22 = 42.' On the right side, add the three '10s' together and get '30,' and '3 + 4 + 5 = 12' and '30 + 12 = 42.' So the sentence is true." As other student-devised strategies are given, the point is made that many computations can be performed mentally and that a certain amount of "power" is gained by those who can learn to do so. The teacher can then ask if anyone can use the pattern of the first three sentences to describe the next sentence in this sequence. The class discussion, where students share their thinking, leads to the following:

$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

$$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$$

The natural question at this point is whether this fourth sentence is true, and here again the students can be asked to try to determine the answer without using paper and pencil or calculator. The author has witnessed several very clever techniques suggested by students. One favorite was given by a fifth-grade student who claimed that all such sentences formed in this way would be true. His reasoning went like this:

The first sentence is easy, so I'll start with the next one. The second sentence starts off with 4, which is 2 twos. Give one of the 2s to the 5 and one of the 2s to the 6, and you get the right side ($7 + 8$).

!Mathematical Expression Omitted^

The third sentence starts off with 9, which is 3 threes. Give a 3 to the 10, a 3 to the 11, and a 3 to the 12, and you get the right side.

!Mathematical Expression Omitted^

He continued in this way until he had shown that his method worked for the four sentences given. The class, and the author, were quite impressed. What would the next sentence in this sequence look like? Would it be a true sentence? Will this strategy be successful there too? This criterion is an attempt to help give value to these questions and these types of discussions.

6. When the strategy you are using isn't working out, you should be willing to try another strategy instead of giving up.

This criterion is probably the most important of all because it focuses on your learning how to think for yourself. It is also one of the hardest because it challenges you to work your way out of tight spots without the help of the teacher or the answer key. Learning how to think means you are learning how to take care of yourself. It means you don't always need someone else to solve your problems for you. Solving your own problems is the most important thing that school can teach you. It offers you a chance to develop a sort of independence and a sort of "power" that will help you learn on your own after school has taught you all it can. I want you to try to think about this criterion every time you get bogged down trying to solve a problem. Try to monitor your growth in this area and see whether you can minimize the number of times you "give up" on a problem. We will have a number of opportunities to talk about your progress throughout our work together.

Discussion and example. This criterion is similar to criterion 2, "getting unstuck," but like the earlier example of the relationship between criterion 4 and criterion 5, it too is more general than its earlier version because it refers to overall strategies rather than isolated facts or procedures. The intent of the criterion is to help students begin to think in terms of overall problem-solving strategies rather than just isolated answers to episodic problems. This example comes from an experience with a seventh-grade class that was working on geometric constructions. The class had been given a homework problem that asked them to construct a 75-degree angle. The next

day, a class discussion showed that several students had solved the problem by constructing two perpendicular lines and bisecting the right angle forming a 45-degree angle. Then by constructing an equilateral triangle, they produced three 60-degree angles. Bisecting one of the angles of the triangle formed a 30-degree angle. Copying adjacent 45-degree and 30-degree angles formed the desired 75-degree angle.

One of the students, who had been unable to solve the problem the night it was assigned, came to class the day following the class discussion and shared the following solution. He said that after he saw what other students had done, he had begun to think about the problem after class. When he figured out his solution, he was excited and wanted to share it with the class. This criterion creates a forum whereby secondary solutions and afterthoughts are an important component of mathematics.

7. You should be able to extend a problem situation by posing additional conditions or questions.

Every problem is determined by the specific conditions that it describes. By changing any of the conditions, you end up with a new and different problem. We will be trying to go beyond solving problems and coming up with answers. Part of our work will be about making up our own questions and problems. After we work a problem, we might ask "What would happen if ...?" This question usually leads to a different problem, and I want you to pay attention to how often we explore it in class and how often you ask it yourself. Albert Einstein once said, "Imagination is more important than knowledge." This criterion is about imagination.

Discussion and example. Problem posing is the ultimate expression of mathematical growth (Bruner 1968; Wirtz 1976). Consider the following problem that was given to a fourth-grade class.

A B +C

AB

Each letter represents a digit (0, 1, 2, ..., 9) and the students were to find the value of C. By trial and error, most students found that $C = 9$ and $A = 1$ and that B could be any other digit. The class discussion intrigued one student, Sophia. She continued to play around with the problem after class to the extent that she removed the restriction that C be a one-digit number. The next day, she shared her findings with the class. It seems that if $A = 7$, and $B = 4$, then

7 4 +C

74

means that $C = 63$. Continuing in this vein, Sophia found that in each case, C would necessarily be a multiple of nine, and in fact would be $9XA$. By extending the problem she found another interesting property of numbers, and she was motivated to share her discovery with the class. More important in the long term, the class discussion that she prompted set a memorable

example to the other students about the potential for posing one's own questions and problems.

These seven criteria, then, form the basis for establishing a classroom environment that advances the notion that "mathematics is a way of thinking." They lay a foundation whereby students can actually practice mathematical thinking while learning it themselves. The criteria go a long way in giving importance to the order and logic on which the field of mathematics is built. They depend on and support students' understandings and the ability to talk about their work in ways that will make sense to others. According to these criteria, mathematics is not seen as being just a chain of rules, each of which produces an answer. In fact, under these conditions, the student is meant to see that one's responsibility for doing a mathematics problem does not end when an answer is obtained. The process of explaining and defending one's strategy is of equal importance. Besides offering many valuable opportunities for students to learn from their own work, whether correct or incorrect, this process is also a critical diagnostic tool the teacher can use to see "inside the student's head" to determine the extent of understanding. The seven criteria support the notion that the strength of knowledge is in the ability not only to answer questions but also to have the sense and confidence to question answers.

Establishing these seven learning criteria alone will not automatically generate instant successful learning in all students. The teacher needs to see that establishing and referring to the learning criteria is an important first step. Given time and attention, the criteria can help students bring more thought and personal responsibility to their learning.

Assessing Students' Progress

We now shift our focus to assessing and grading students. In this area, the role of the teacher is to gather information that describes how students respond to each of these criteria; organize what is observed; and turn it into usable, accurate assessment data. The following discussion contains some ideas that have been found useful in this effort.

Whenever working or explaining finished work, the student supplies the teacher with information that can be used to gauge growth along at least one of the criteria. In the beginning, these observations can be recorded on the record sheet shown in table 2. After a while, as the assessment process is internalized by the teacher, most assessment data are more easily remembered and organized without having to record them. At any rate, in the beginning the recording sheet can be used as follows:

- a. To keep an unmarked original record form, make a photocopy of it and refile the original for future use.
- b. Write each student's name on the copy.
- c. Make one copy of this record for each month assessment data are collected.
- d. Equate each number at the top of the form with one of the criteria. For example:

1 = Everything you do in mathematics should make sense to you. (Understands)

2 = Whenever you get stuck, you should be able to use what you know to get yourself unstuck. (Gets unstuck)

3 = You should be able to identify errors in answers in the use of materials and in thinking. (Spots errors)

4 = Whenever you do a computation, you should use a minimum of counting. (Minimum of counting)

5 = You should be able to perform calculations with a minimum of rote pencil-paper TABULAR DATA OMITTED computations. (Minimum paper-pencil use)

6 = When the strategy you are using isn't working out, you should be willing to try another strategy instead of giving up. (Perseverance)

7 = You should be able to extend a problem situation by posing additional conditions or questions. (Problem posing)

e. Select a grading and recording system to use on the student data that will be collected.

Letter grades, such as the traditional A-F, or others, such as the one offered here, can be used. If you prefer, the numbers 1-5 could be used in place of the letters and might be considered when a numerical average is desired. Several suggestions are given in table 3. Whichever code is selected, the teacher records the grade or score under the appropriate criterion when a student provides information to the teacher. For example, Su-Lin is observed working with another student, Hans. The teacher overhears the following conversation:

Su-Lin: "Wait a minute, that's not right. Six times eight isn't fifty-six."

Hans: "Yes it is."

Su-Lin: "No it's not. I'll bet you it's not."

Hans: "Okay, let's get the grid and masks" (the materials being used at the time).

They get the grid and masks, and Hans discovers that he's made a mistake. The teacher has gained valuable assessment information on both students. Su-Lin spotted an error and called it to Hans's attention, so the teacher can record an "E" or "V" under the number "3" next to her name (using the middle coding system). Hans was able to focus on the question raised by Su-Lin and had some sense of how to find the correct answer. In doing so, he found his error and corrected it. The teacher needs to decide whether this occurrence was an example of getting unstuck or finding an error, or both, but it is useful information no matter where it is recorded. It should also be noted that had Hans not suggested the grid and masks as a way of finding the correct answer, or if neither of them spotted the error, this information would also have been useful and could

have been translated into a "B" or "N" for both. The situation given in the foregoing dialogue would be worthy of a class discussion to help students better understand how such interactions can be used to assess learning. It might be introduced to the class like this:

"While I was walking around during your work time, I saw Su-Lin and Hans discussing a difference of opinion about one of the problems. Su-Lin and Hans, are you willing to tell the class what happened?" After the situation is described, ask the class whether they can tell which of the learning criteria was demonstrated in that example, and then go on to tell Su-Lin and Hans that they need to think about taking credit for the way in which they handled it. (See Greenwood 1991, 1992[^] for more examples.)

The teacher should make an effort to collect information for each criterion on every student at least once per month. Therefore, copies of the record sheet are needed for each month of use. Often, if enough information hasn't been gathered for, say, Josef and the class is approaching the end of the month, the teacher may tell him that he will be asked to share some of his work at the front of the class the next day (or soon). It is important to give students advance notice when they will be asked to do something that involves personal risk (Newton 1985). It helps them prepare their focus and they are TABULAR DATA OMITTED not jolted with surprise when called on. In such situations, the teacher can also make a note to observe these students more closely. As time goes on, the teacher gets a lot better at pacing observations and spreading them around more evenly throughout the month.

At the end of each month the students can be asked to evaluate themselves according to the same seven criteria. By doing so, the students take an active part in monitoring their own growth and have a chance to compare their evaluative statements with the teacher's. Table 4 is a copy of the general form used with students.

On Monday or Tuesday of the last week of each month, the form is distributed to each student. The students are asked to consider each criterion carefully and to grade themselves, marking in the left column, according to how they think they've done over the course of the month. They are asked to use the same grading code that will be used by the teacher. The students can use the suggested code given previously in their self-evaluations. The code that follows has also been used with students because many find it easier to interpret:

A = Always M = Most of the time O = Occasionally S = Seldom N = Not at all

After sufficient time, students' self-evaluations are collected. The teacher then goes over each student's form carefully and evaluates it using the same grade codes, marking in the rightmost column. In the beginning, it is not unusual for students to be more critical of themselves and more negative with their own self-given scores than the assessments given by the teacher. Whenever these discrepancies occur, the student can be asked to try to describe how the teacher's grade was determined. That is, what information did the teacher use to assess the student's growth as being greater than that given by the student? The question is more than rhetorical, since it asks the student to reflect on how she or he is perceived by others; and since the teacher's grade is generally higher, it offers the students a chance to think about how she or he has grown. As time goes on, students develop a better sense for assessing themselves and seem to keep the

criteria more in mind when they are working. Again over time, students lose the biting edge of self-criticism and become more kind to themselves. When the teacher finishes writing responses on the forms, they are returned to the students with the request that they be taken home and shown to their parents. They are to be returned the next day with the parent's signature and are kept until the end of the grading period. These informal assessments are intended to supplement the formal grades that are sent home at the end of each grading period. This type of monthly communication is vitally important in monitoring and reporting students' growth. It has been found that this total assessment program is effective for several reasons:

- * It gives the students a chance to grade themselves, which is critical to developing a sense of personal involvement and personal responsibility in one's learning.
- * It gives students an opportunity to compare their evaluative judgment with someone else's. This type of "reality check" helps solidify the student's "outside-in" view of self.
- * The learning criteria become more familiar, more important, and more useful to the students as they learn how to apply the criteria both to learning and to the assessment of learning.
- * Students are not surprised when grades come out. They are much more a part of the grading process, and they begin to see it as a "consequence" of their efforts during the previous days and weeks.

Suggestions for Grading

The following discussion is difficult to broach because of the personally sensitive nature of grading students. It is also difficult because of the laudable work done recently to move assessment toward holistic parameters rather than reduce them to simplistic letter grades of the past (Stenmark 1989). The suggestions offered here are not intended in any way as a criticism of that work, nor as implied support for letter grades and grade-point averages. The intent rather is to offer helpful suggestions to teachers who find themselves struggling with the problems of transition that will help broaden the concept of grades so that they better reflect students' growth in mathematical power and mathematical thinking and place less emphasis on procedural knowledge that is the domain of tests and quizzes.

With this intent in mind, the process of translating the foregoing discussion into recorded grades is the next step in expanding student assessment to reflect these new dimensions. It is important that students' grades reflect all the areas that have been the focus of the seven learning criteria given previously. Grades that are solely dependent on test scores and on the completion of homework assignments simply do not capture the quality of learning that is proposed and intended by the NCTM's Curriculum and Evaluation Standards (1989). The teacher can quantify each of the seven learning criteria by using numbers instead of letter grades, find their sum, and add the test scores to this total. Dividing this accumulated total by 8 (the seven criteria plus the average test score), provides the teacher a meaningful, more complete measure of the student. When weighing each area somewhat equally, a composite grade might then be determined by averaging the eight criteria. Several alternatives in weighing are given in table 5.

TABLE 5

Suggestions for weighting the different criteria

| Ex. 1 weight | Ex. 2 weight | Ex. 3 weight | Criterion |
|-----------------|-----------------|-----------------|-------------------------------|
| 13% | 15% | 10% | Understanding |
| 13% | 10% | 15% | Getting unstuck |
| 12% | 10% | 10% | Spotting errors |
| 12% | 10% | 15% | Using minimum of counting |
| 13% | 20% | 15% | Using minimum of paper-pencil |
| 12% | 10% | 15% | Perseverance |
| 12% | 10% | 10% | Problem posing |
| 13% | 15% | 10% | Tests and quizzes |
| 100% | 100% | 100% | Total grade |

Responses from parents, students, and administrators have shown that such a grade reflects a more total picture of the student's progress during the grading period. It also supplies valuable information for parents in their attempts to help students at home.

Summary

Implementing the NCTM's curriculum standards and professional teaching standards (1989, 1991) requires operational definitions for such terms as mathematical power and mathematical thinking. Without a workable process for defining our work in teaching and assessing these noble goals, we stand exposed to the close scrutiny of those more interested in test scores as a means of measuring students' growth. The ideas and experiences presented in this article are shared as an attempt to contribute to this effort. They are based on the use of seven learning criteria that help put into operation the concepts of mathematical power and mathematical thinking. The criteria can be used across grade levels and for all mathematical topics as a basis for planning, teaching, and assessing students' growth.

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TABLE 1

Criteria for mathematical thinking

1. Everything you do in mathematics should make sense to you.
2. Whenever you get stuck, you should be able to use what you know to get yourself unstuck.
3. You should be able to identify errors in answers, in the use of materials, and in thinking.
4. Whenever you do a computation, you should use a minimum of counting.
5. You should be able to perform calculations with a minimum of rote pencil-paper computations.
6. When the strategy you are using isn't working, you should be willing to try another strategy instead of giving up.
7. You should be able to extend, or change, a problem situation by posing additional conditions or questions.

Jay Greenwood teaches at Georgia Southwestern College, Americus, GA 31709, and also works with students at Sumter County Middle School

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